Laminar mixed convection in vertical parallel plates channels with symmetric UWT boundary conditions

M.Sc. Thesis

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- the identification of the correct *buoyancy parameter* Gr_{D_h}/Re_{D_h}, obtained from the dimensionless form of governing equations
- a comprehensive analysis of the friction and heat transfer coefficients ($f(y)Re_{D_h}$ and $Nu(y)$), velocity and temperature boundary layers, performed with the *Cast3m* code
- the verification of the validity of the *Reynolds analogy* also in mixed convection configuration, supported by an approach based on the ratio between mixed and forced velocity boundary layers

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- the verification of the validity of the *Reynolds analogy* also in mixed convection configuration, supported by an approach based on the ratio between mixed and forced velocity boundary layers
- the definition of a novel diagram of the flow reversal occurrence, in the $(Gr_{D_h}/Re_{D_h})_{crit}$ vs Pe_{D_h} coordinates, using Pr as parameter

Governing equations, geometry and boundary conditions

The Boussinesq's equations

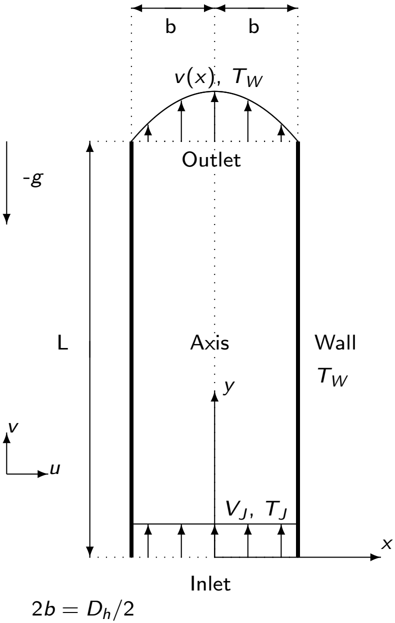
The 2D governing equations are written in the elliptic form, under the *Boussinesq approximation* hypotheses:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu_0\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{1}{\varrho_0}\frac{\partial p'}{\partial x} \quad (2)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu_0\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{1}{\varrho_0}\frac{\partial p'}{\partial y} - \varrho_0\beta_0(T - T_0)g \quad (3)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_0\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \quad (4)$$



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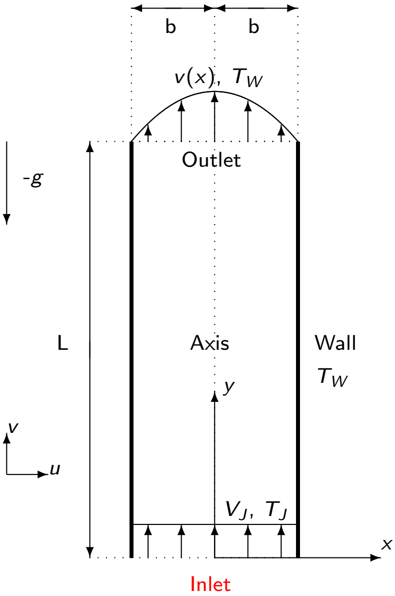

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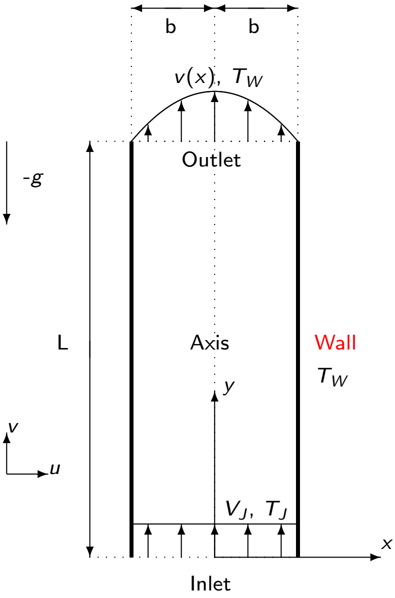
- Inlet: $0 \le x < b$, $y = 0$:

$$u = 0 \qquad T = T_J$$
$$v = V_J$$



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$$u = 0 \qquad T = T_W$$
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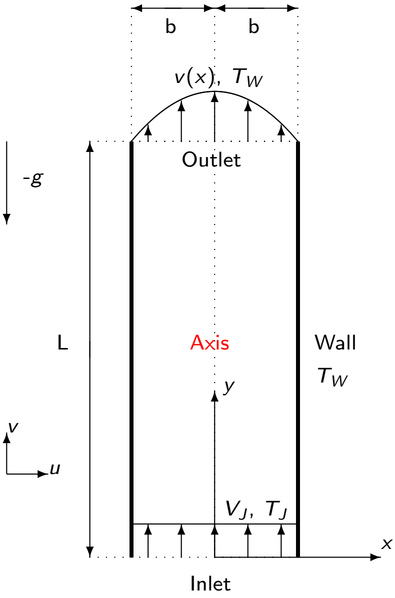

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- Axis: $x = 0$, $0 \leq y \leq L$:

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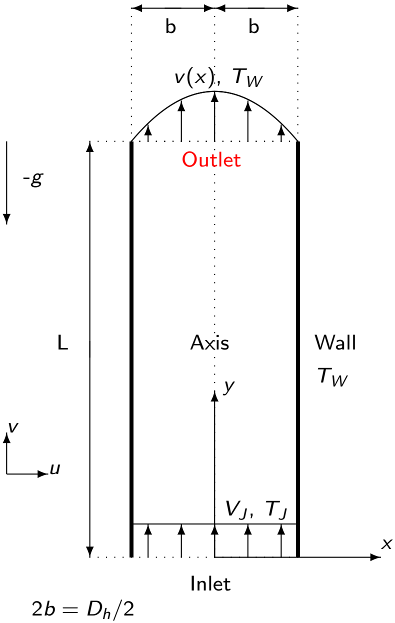

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- Outlet: $0 \leq x < b,\ y = L$:

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial T}{\partial y} = 0$$
$$\frac{\partial v}{\partial y} = 0 \qquad p = p_0$$

The identification of length scales for mixed convection in duct flows

The scale analysis

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$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad \frac{u}{D_h} \sim \frac{V_J}{L_y} \tag{6}$$

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The *energy conservation equation* gives:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_0\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \quad \Rightarrow \quad V_J\frac{\Delta T}{L_y} \sim \alpha_0\frac{\Delta T}{D_h^2}, \cancel{\alpha_0\frac{\Delta T}{L_y^2}} \tag{7}$$

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By expliciting D_h from eq. 7:

$$D_h \sim L_y Re_{L_y}^{-1/2} Pr^{-1/2} \tag{8}$$

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More useful informations can be achieved from the *phenomenological analysis*.

The phenomenological analysis

If we suppose to introduce a finite Heavyside temperature disturbance in the duct flow, after a time $\tau_{dif} = \frac{1}{\alpha_0}\left(\frac{D_h}{2}\right)^2$, the heat transfer by means of molecular diffusion has affected a fluid portion inside a radius equal to the distance between the plates $2b = D_h/2$.

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By recalling the Reynolds number based on D_h, that is $Re_{D_h} = \frac{V_J D_h}{\nu_0}$, we write:

$$\ell_{adv} = \left(\frac{D_h}{4}\right)\left(\frac{V_J D_h}{\nu_0}\right)\left(\frac{\nu_0}{\alpha_0}\right) \qquad \Rightarrow \qquad \ell_{adv} = 0.25 D_h Re_{D_h} Pr \tag{12}$$

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By virtue of the meaning of the advective length ℓ_{adv}, we define it as the *asymptotic thermal length* L_∞^{th} (all the figures in the following are based on this length):

$$L_\infty^{th} = 0.25 D_h Re_{D_h} Pr \tag{13}$$

and again, we identify the *longitudinal reference length* L_y, as:

$$L_y = D_h Re_{D_h} Pr \tag{14}$$

The dimesionless form of the governing equations

The dimensionless governing equations

After the definition of the reference length scales $L_x = D_h$ and $L_y = D_h Re_{D_h} Pr$, we can write the dimensionless form of the governing equations, based on the *asymptotic thermal length* L_∞^{th}:

$$X = \frac{x}{L_x} = \frac{x}{D_h} \quad \text{and} \quad Y = \frac{y}{L_y} = \frac{y}{D_h Re_{D_h} Pr} \tag{15}$$

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The continuity equation gives the expression of the transversal velocity u.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad \frac{u}{D_h} \sim \frac{V_J}{D_h Re_{D_h} Pr} \quad \text{that is} \quad u \sim \frac{\alpha_0}{D_h} \tag{16}$$

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Dimensionless velocities, temperature and pressure are:

$$\tilde{u} = \frac{u D_h}{\alpha_0} \qquad \tilde{v} = \frac{v}{V_J} \qquad \Theta = \frac{T - T_J}{T_W - T_J} \qquad \tilde{p}' = \frac{p'}{\varrho_0 V_J^2} \tag{17}$$

The dimensionless governing equations (2)

Finally, the dimensionless set of equations for *laminar mixed convection* in *duct flows* is:

$$\frac{\partial \tilde{u}}{\partial X} + \frac{\partial \tilde{v}}{\partial Y} = 0 \quad (18)$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial X} + \tilde{v}\frac{\partial \tilde{u}}{\partial Y} = Pr\frac{\partial^2 \tilde{u}}{\partial X^2} + \left(\frac{1}{Re_{D_h}^2 Pr}\right)\frac{\partial^2 \tilde{u}}{\partial Y^2} - \left(Re_{D_h}^2 Pr^2\right)\frac{\partial \tilde{p}'}{\partial X} \quad (19)$$

$$\tilde{u}\frac{\partial \tilde{v}}{\partial X} + \tilde{v}\frac{\partial \tilde{v}}{\partial Y} = Pr\frac{\partial^2 \tilde{v}}{\partial X^2} + \left(\frac{1}{Re_{D_h}^2 Pr}\right)\frac{\partial^2 \tilde{v}}{\partial Y^2} - \frac{\partial \tilde{p}'}{\partial Y} - \left(\frac{Gr_{D_h}}{Re_{D_h}}\right)Pr\,\Theta \quad (20)$$

$$\tilde{u}\frac{\partial \Theta}{\partial X} + \tilde{v}\frac{\partial \Theta}{\partial Y} = \frac{\partial^2 \Theta}{\partial X^2} + \left(\frac{1}{Re_{D_h}^2 Pr^2}\right)\frac{\partial^2 \Theta}{\partial Y^2} \quad (21)$$

The buoyancy parameter Gr_{D_h}/Re_{D_h} represents the amount of the natural convection with respect to the forced convection, or equivalently the ratio between the natural to forced velocities.

$$\frac{Gr_{D_h}}{Re_{D_h}} = \left(\frac{g\beta\Delta T D_h^3}{\nu^2}\right)\left(\frac{\nu}{V_J D_h}\right) = \left(\frac{g\beta\Delta T D_h^2}{\nu}\right)\frac{1}{V_J} = \frac{V_{nat}}{V_J} \quad (22)$$

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- Dimensionless form based on the *hydraulic diameter* D_h:

$$L_x = D_h \qquad \text{and} \qquad L_y = D_h \tag{23}$$

  This leads to a different buoyancy parameter $Gr_{D_h}/Re_{D_h}^2 = Ri_{D_h}$, which is appropriate only for *boundary-layer flows*, but not for *duct flows*.

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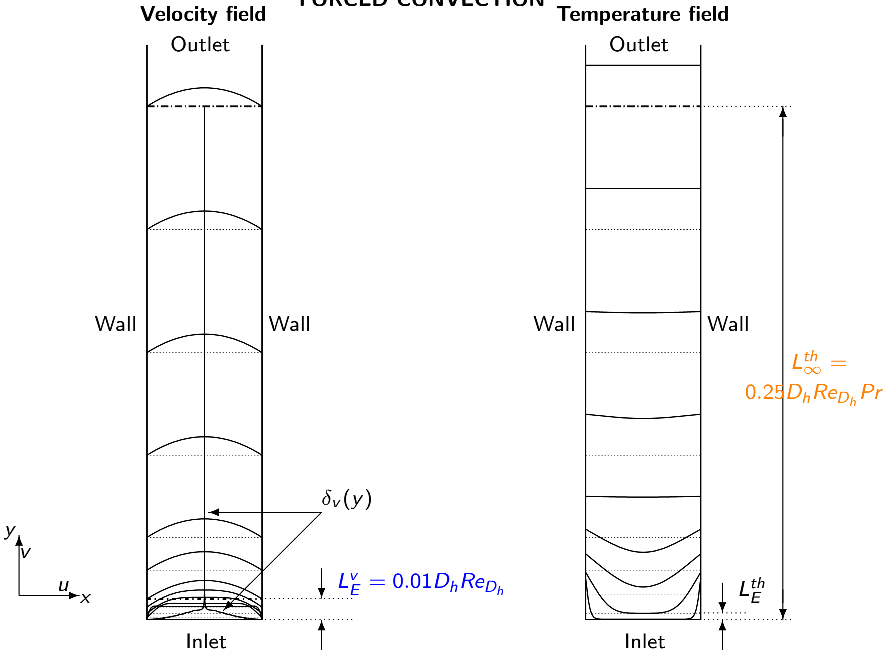
- Dimensionless form based on the *velocity entry length* L_E^v([1]):

$$L_x = D_h \qquad \text{and} \qquad L_y = D_h Re_{D_h} \tag{24}$$

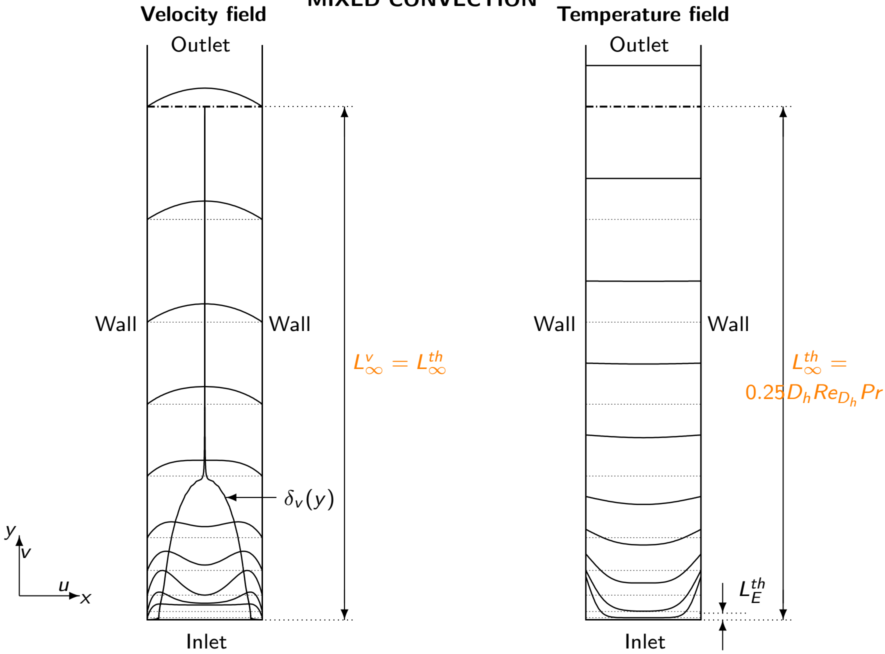
This approach takes into account the developing only of the velocity field, as it were not coupled with temperature (forced convection); although it leads to the correct buoyancy parameter Gr_{D_h}/Re_{D_h} for mixed convection, the subsequent evaluation of the fully developed flow length is erroneous (compare L_E^v vs L_∞^{th} in the following figures).

[1] The *velocity entry length* L_E^v is the distance from the inlet section where the *velocity boundary layers* δ_v interact on the axis, and the velocity achieves the 99% of the fully developed parabolic profile.

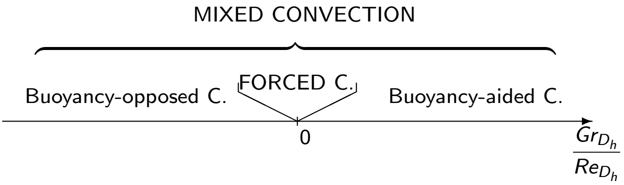
FORCED CONVECTION



MIXED CONVECTION



Relation between forced and mixed convection



In mixed convection, the flow is subject to a temperature gradient and to a buoyancy force proportional to density variation (due to the thermal and gravitational fields).
The ligther density layers assume the ascensional motion respect to the colder ones which flow downward; depending on the sign of the temperature difference ΔT, this driving force can act in the same direction of inertial forces (BA convection) or in opposed direction (BO convection).

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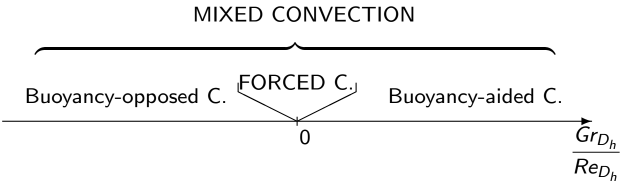

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Aided case, BA: the upward buoyancy forces recall and accelerate the flow in a squeezed region between the boundary layer and an impairment velocity zone which appears in the center of the duct, due to the conservation of the mass balance.

Opposed case, BO: the downward buoyancy forces counteract the ascending motion in vicinity of the walls, and require the flow to deflect and to accelerate in the bulk region.

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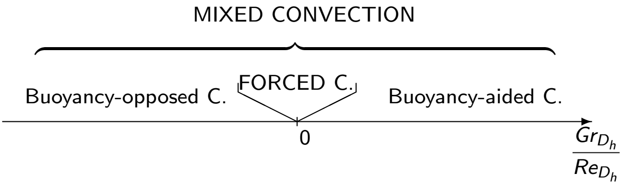

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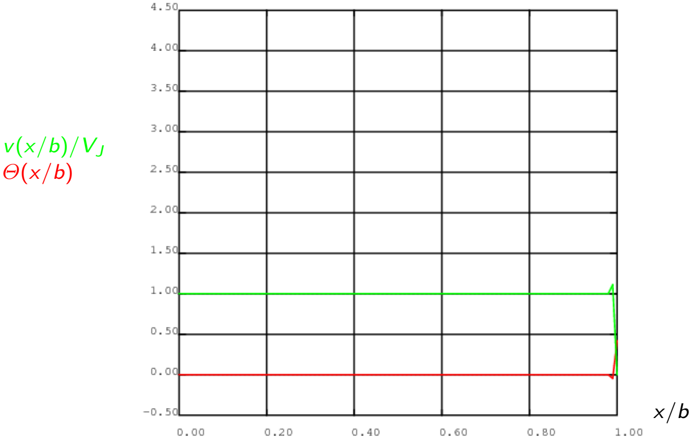
Opposed case, BO: the downward buoyancy forces counteract the ascending motion in vicinity of the walls, and require the flow to deflect and to accelerate in the bulk region.

Flow reversal phenomenon occurs both in BA and BO cases, when buoyancy intensity is larger than the critical value, identified by the $(Gr_{D_h}/Re_{D_h})_{crit}$ value. Flow reversal means that, locally, the fluid flows in the opposite direction respect to the imposed velocity V_J.

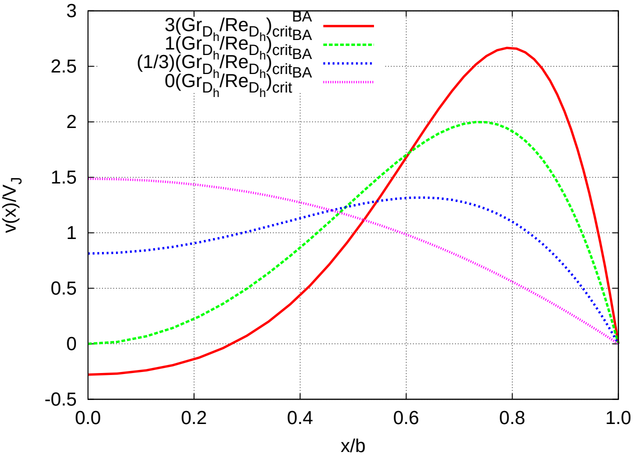
Buoyancy-aided convection

The critical buoyancy parameter in aided convection has been numerically found equal to $(Gr_{D_h}/Re_{D_h})^{BA}_{crit} = 2400$, conferming the value obtained by Desrayaud and Lauriat (2009).
We report the velocity (I) and temperature (I) profiles for different sections (from the inlet to the outlet), for the case $3(Gr_{D_h}/Re_{D_h})^{BA}_{crit}$, with the occurrence of flow reversal in the bulk region.



The velocity field

In buoyancy-aided convection, the velocity gradient at the wall $\left(\frac{\partial v(x)}{\partial x}\right)_W$ is enhanced due to the acceleration of the hot and lighter layers (buoyancy forces acting in the same direction with respect to the upward flow).

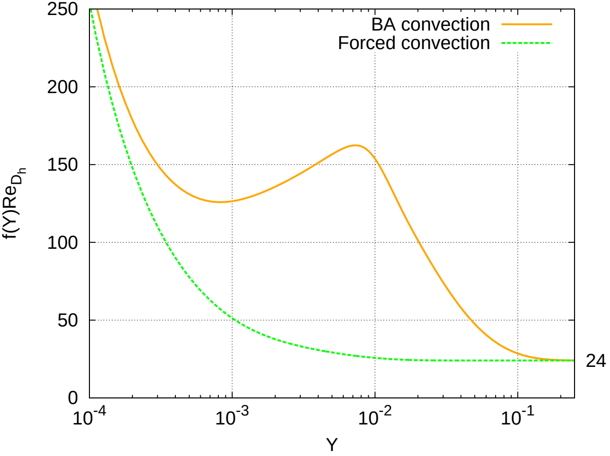


Friction factor times Reynolds number

The friction factor $f(y)$ is subsequently enhanced in buoyancy-aided convection, because:

$$f(y) = \frac{2\tau_W(y)}{\varrho_0 V_J^2} = \left(\frac{2\nu_0}{V_J^2}\right) \frac{\partial v(x,y)}{\partial x}\bigg|_W \tag{25}$$

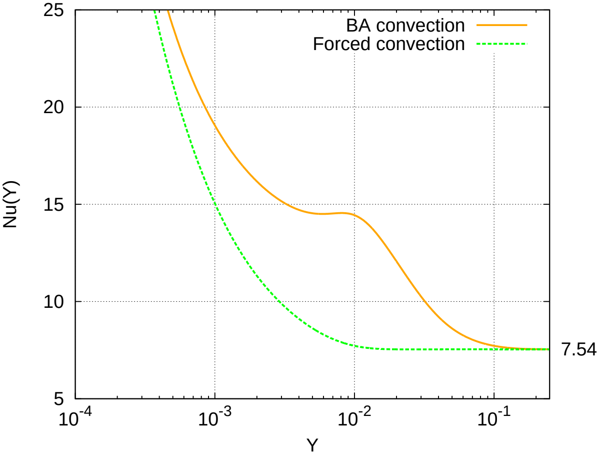
It is here reported the plot of the local friction factor times Reynolds number $f(y)Re_{D_h}$ for the forced and BA convection cases.



Nusselt Number

Due to the flow acceleration, the colder advective region (velocity peak) is in intimate shearing contact with the hotter transversal diffusive region at the wall. The convective heat coefficient $h(y)$, and the Nusselt number $Nu(y) = h(y)D_h/\kappa_0$, are therefore enhanced.

$$Nu(y) = \frac{hD_h}{\kappa_0} = \frac{D_h \left.\frac{\partial T(x,y)}{\partial x}\right|_W}{(T_W - T_b(y))} \quad \text{where } T_b(y) \text{ is the bulk mean temperature} \qquad (26)$$

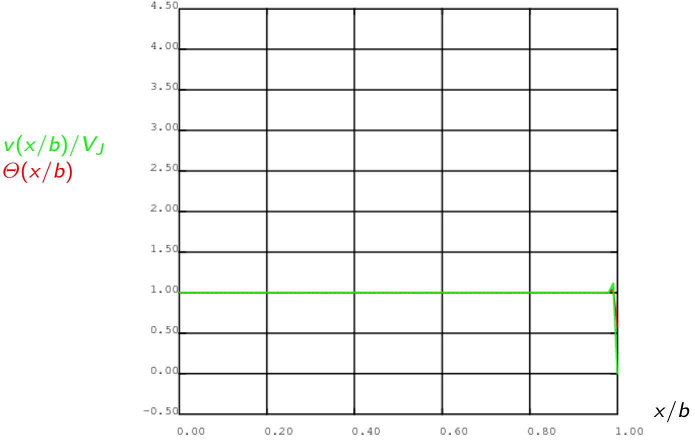


Mixed convection:
Buoyancy-opposed convection

Buoyancy-opposed convection

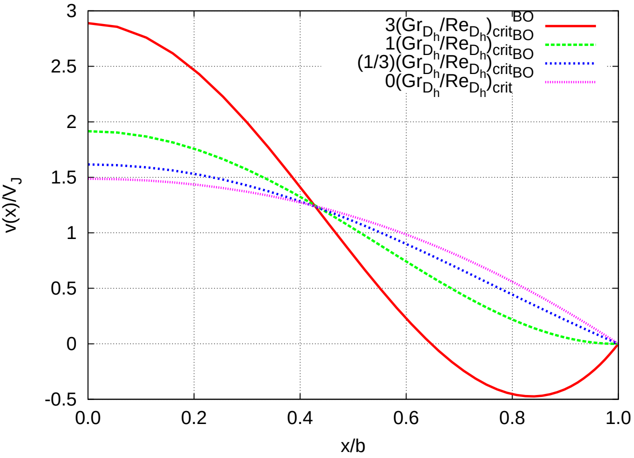
The critical buoyancy parameter in opposed convection has been numerically found equal to $(Gr_{D_h}/Re_{D_h})^{BO}_{crit} = -465$; no comparison with previous works is possible, with the exception of Ingham (1988) which uses the parabolic formulation.

We report the velocity (|) and temperature (|) profiles for different sections (from the inlet to the outlet), for the case $3(Gr_{D_h}/Re_{D_h})^{BO}_{crit}$, with the occurrence of flow reversal in the wall region.



The velocity field

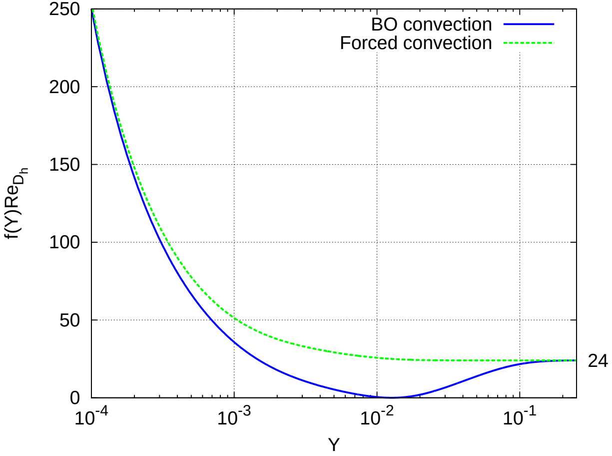
In buoyancy-aided convection, the velocity gradient at the wall $\left(\frac{\partial v(x)}{\partial x}\right)_W$ is smooth due to the impairment of the cold and heavier layers (buoyancy forces acting in the opposite direction with respect to the upward flow).



Friction factor times Reynolds number

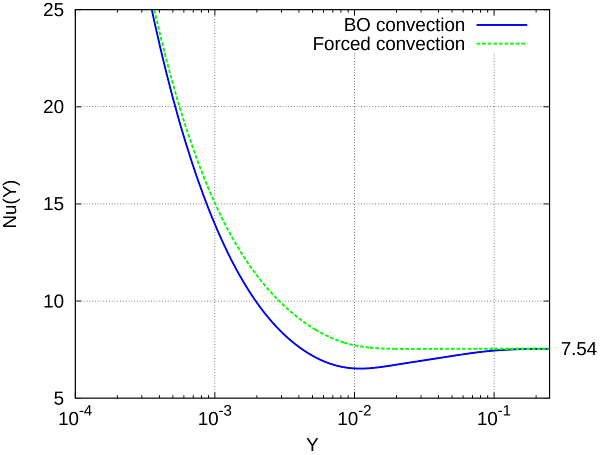
The friction factor f is impaired in BO convection.

It is here reported the plot of the local friction factor times Reynolds number $f(y)Re_{D_h}$ for the forced and BO convection cases.



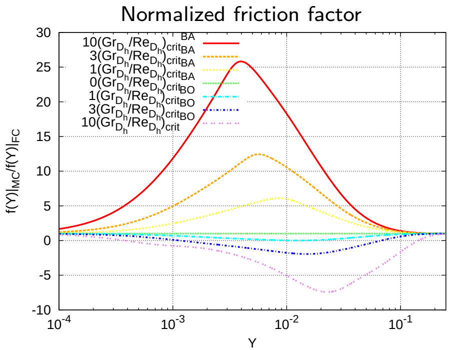
Nusselt Number

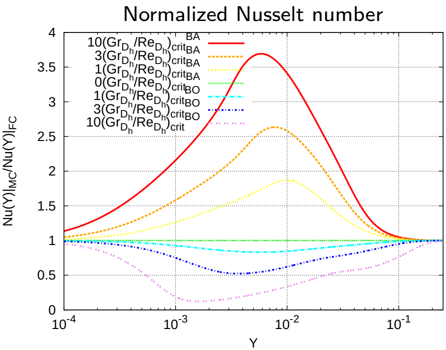
Due to the flow impairment, it occurs that a large thermal diffusion region is present near to the wall. The convective heat exchange, evaluated by the $h(y)$ coefficient, or by the Nusselt number $Nu(y) = h(y)D_h/\kappa_0$, is therefore impaired.



BA and BO comparison

Local friction factor times Reynolds and Nusselt number





The BA curves (III) for $f(y)Re_{D_h}$ and $Nu(y)$ are similar, due to the validity of the *Reynolds analogy*, verified for mixed convection in the present work with a proposed correlation based on the velocity boundary layers ratio.

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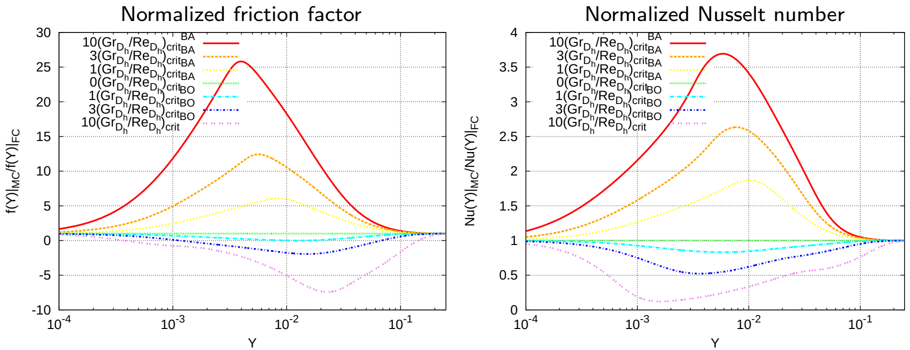

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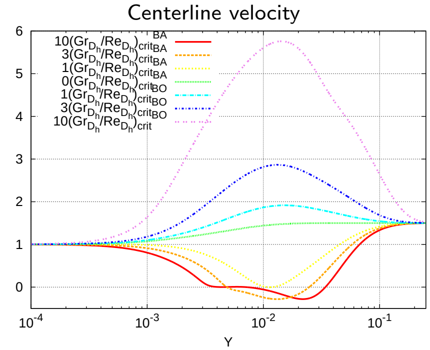
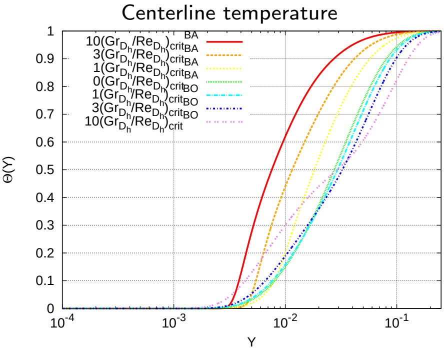
For the BO curves (III) the minimum value of the Nusselt ratio (local minimum heat exchange) is located at those Y the correspondent friction factor ratio has an inflection point (local minumum wall friction); on the other hand, the friction factor ratio has a minimum value (local maximum wall friction) occurs where the correspondent Nusselt ratio has an inflection point (local maximum heat exchange).

Centerline velocity and temperature





The BA convection cases (III) show an impairment of the axial velocity, which reaches 0 for the critical buoyancy parameter value (I), and becomes negative for supercritical values. Note the presence of the stagnation region, i.e., $v = 0$ (I).
The dimensionless centerline temperature $\Theta(Y)$ is globally developed in a shorter distance.

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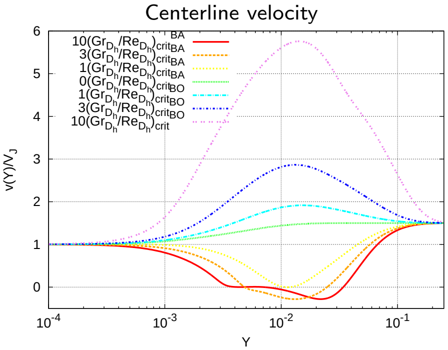

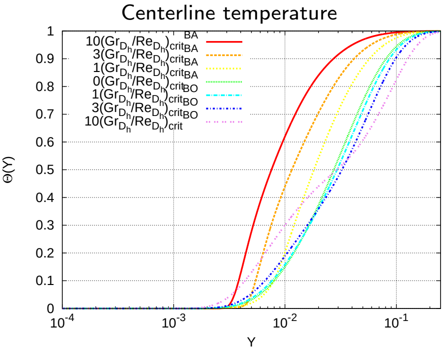

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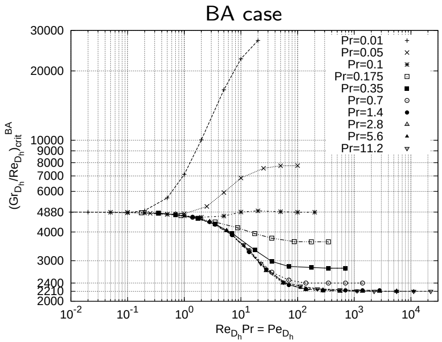
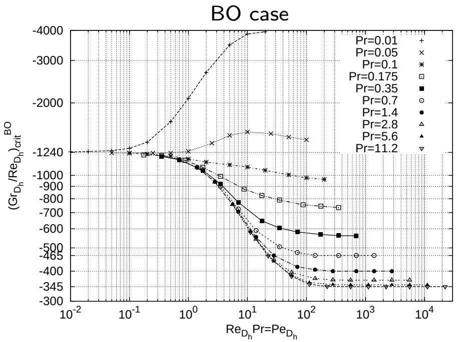
In the BO convection (III), the presence of recirculation cell, near to the walls, constraints the flow to accelerate in the bulk, thus the axial velocity reaches high values in the developing region. Axial temperature (I) is affected at Y coordinate smaller than for BA convection. This is due the advective recirculation flow, which takes place from the wall to the bulk, but this advance does not prevent the thermal developing from being globally retarded and impaired.

Flow reversal regime diagram

Flow reversal regime diagram

The flow reversal regime diagram reports the critical buoyancy parameter values, as a function of the Pe_{D_h} and using Pr as a parameter.





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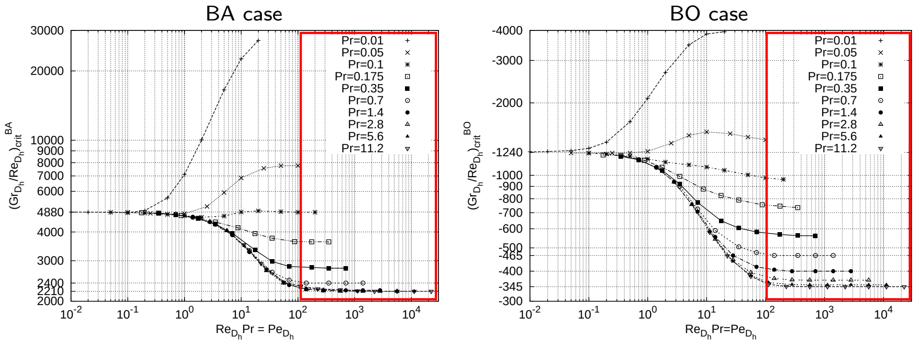

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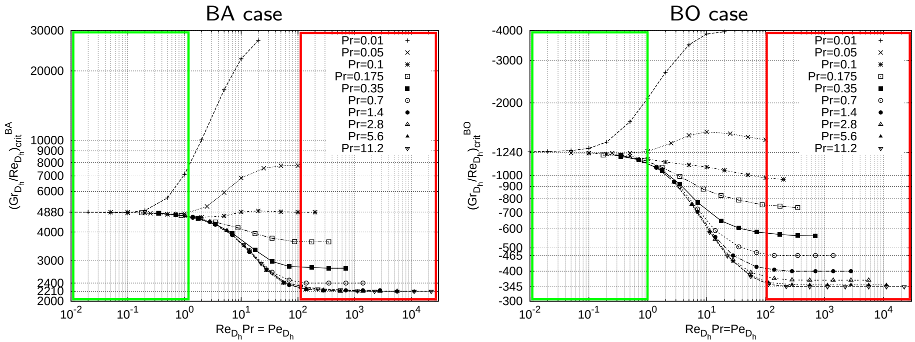

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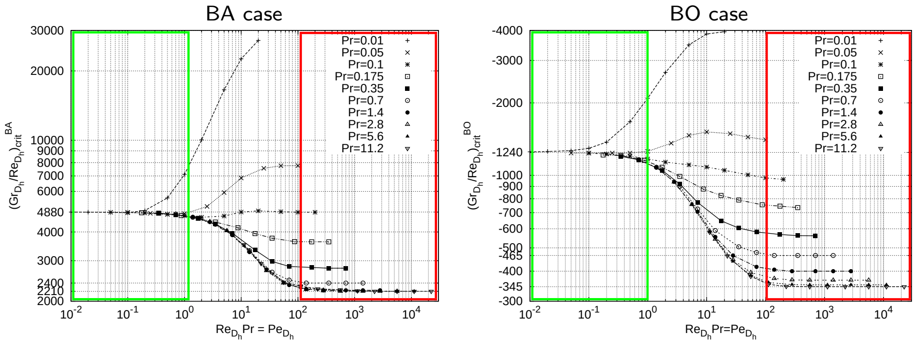

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- a transition zone for intermediate values.

The Reynolds analogy

The Reynolds analogy in forced convection

The relation between the fluid friction and heat transfer, for $Pr \sim 1$ fluids, is given as:

$$(fRe_{D_h})_{FC} = 3.183(Nu)_{FC} \quad \text{where} \quad 3.183 = \frac{f_\infty Re_{D_h}}{Nu_\infty} = \frac{24}{7.54} \tag{27}$$

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$$h = \frac{\dot{q}''}{\Delta T_b} = \frac{\kappa_0 \left.\frac{\partial T}{\partial x}\right|_W}{T_W - T_b} \sim \frac{\kappa_0 \frac{\Delta T}{\delta_T}}{\Delta T} = \frac{\kappa_0}{\delta_T} \tag{29}$$

It follows that:

$$fRe_{D_h} \sim \frac{D_h}{\delta_v} \quad \text{and} \quad Nu \sim \frac{D_h}{\delta_T} \tag{30}$$

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According to the Reynolds analogy theory, $(fRe_{D_h})_{FC} \sim (Nu)_{FC}$ implies that:

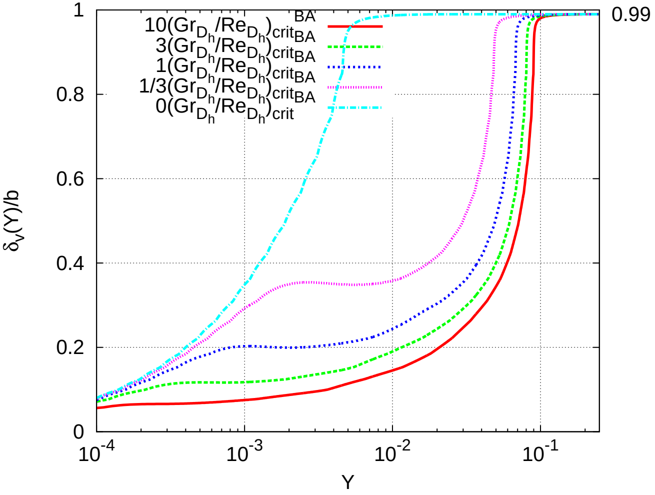
$$\left(\frac{1}{\delta_v}\right)_{FC} \sim \left(\frac{1}{\delta_T}\right)_{FC} \tag{31}$$

Therefore, the Reynolds analogy in forced convection is fundamentally based on the analogy between velocity and temperature boundary layers.

The Reynolds analogy in mixed convection

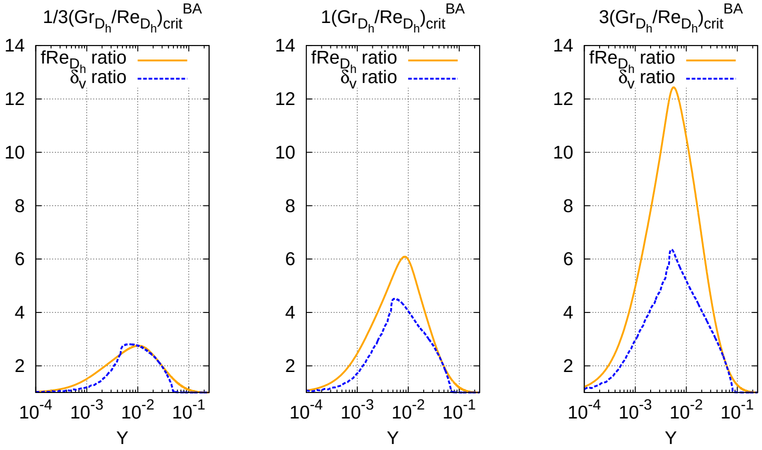
From the analysis of the velocity boundary layer, we made the following hypothesis on the dependency of the friction factor:

$$\frac{(fRe_{D_h})_{MC}}{(fRe_{D_h})_{FC}} \sim \frac{(\delta_v)_{FC}}{(\delta_v)_{MC}} \tag{32}$$



The Reynolds analogy in mixed convection (2)

And then we verified $\dfrac{(fRe_{D_h})_{MC}}{(fRe_{D_h})_{FC}} \sim \dfrac{(\delta_v)_{FC}}{(\delta_v)_{MC}}$:



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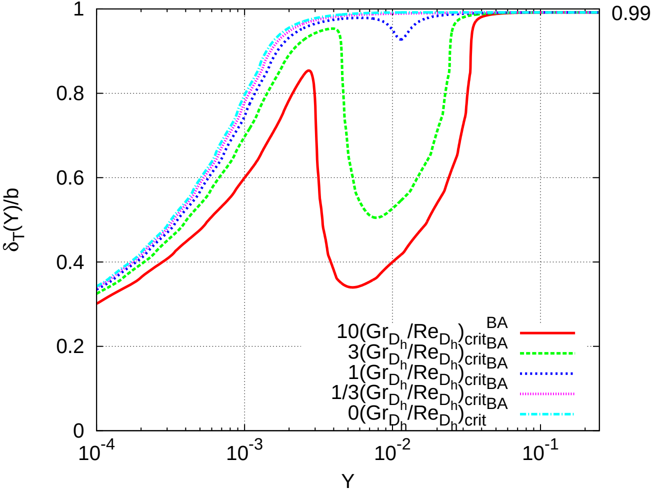
indeed, for undercritical buoyancy, the temperature boundary δ_T remains similar to the forced case (||), whereas for critical and supercritical buoyancy, it almost concides with the recirculation cell bounds (|||).



The Reynolds analogy in mixed convection (4)

Therefore, from the *forced convection Reynolds analogy* equation $(fRe_{D_h})_{FC} = 3.183(Nu)_{FC}$, and the correlation for the friction factor ratio $\frac{(fRe_{D_h})_{MC}}{(fRe_{D_h})_{FC}} \sim \frac{(\delta_v)_{FC}}{(\delta_v)_{MC}}$, the following correlation for the *mixed convection Reynolds analogy* is proposed:

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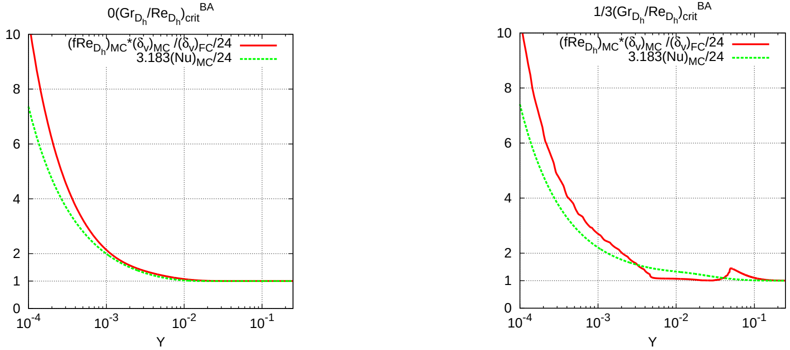
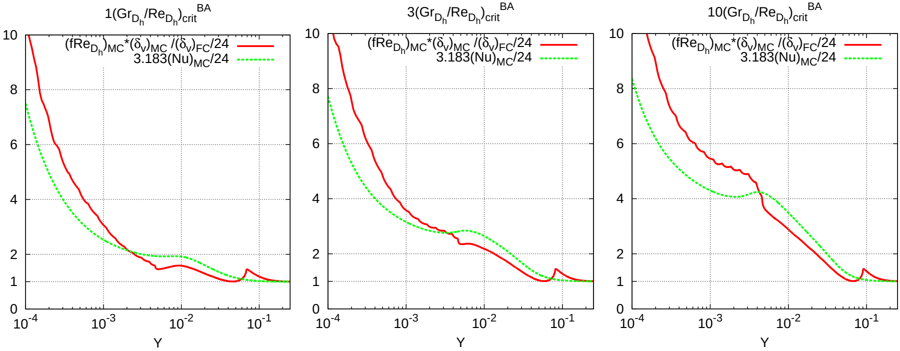
The following figures report the verification of the Reynolds analogy for mixed convection using the proposed correlation based on the velocity boundary layers ratio, at different values of buoyancy:



The Reynolds analogy in mixed convection (5)



Conclusions

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- A series of numerical simulations with *Cast3M* code is performed for the buoyancy-aided and buoyancy-opposed convection: a comprehensive analysis of the friction and heat transfer coefficients, velocity and temperature fields, has led to a deeper undestanding of the phenomena, and of the flow reversal characteristics, as a function of the buoyancy parameter.

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- A novel diagram of the flow reversal occurrence is reported, in the $(Gr_{D_h}/Re_{D_h})_{crit}$ vs Pe_{D_h} coordinates, using Pr as parameter.

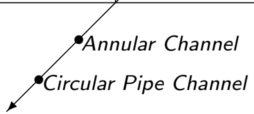
Work scenario

This fundamental study actually is orientated at a theorical and phenomenological studies frame, whose main objective is the *turbulent mixed convection flow*: in fact, almost all nuclear convective applications involve turbulent flows.

The UWT boundary condition has been choosed as starting point because it offers more possibilities of future developments, in the direction of nuclear waste cooling (UHF) but also in the direction of containment atmoshpere mixing (UWT).

Different geometries could be addressed in future works.

Parallel Plates	**Heat Transfer**		**Heat and Mass Transfer**
Channel	**UHF**	**UWT**	**UWT**
LAMINAR FLOW	Nuclear wastes cooling	Present work	Containment atmoshpere mixing
TURBULENT FLOW	Nuclear wastes cooling	Future work	Containment atmoshpere mixing



Laminar mixed convection in vertical parallel plates channels with symmetric UWT boundary conditions

M.Sc. Thesis

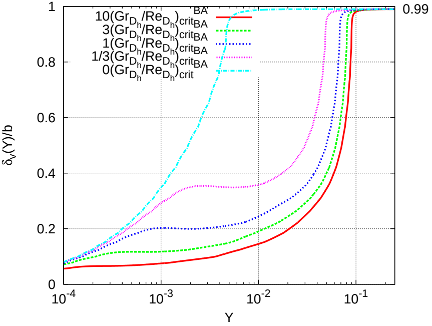
Marco Pieri

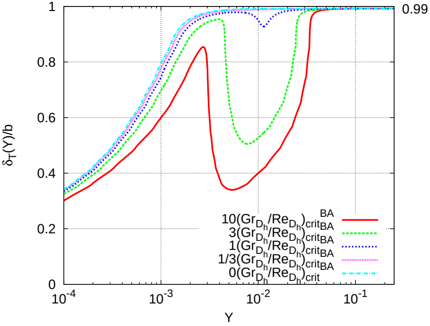
UNIVERSITÀ di PISA

CEA Saclay - DEN/DANS/DM2S/SFME/LTMF

November 26, 2009

Velocity and temperature boundary layers





Hydrodynamic

Buoyancy driven flow governing equations: the Boussinesq's approximation

Buoyancy driven flow governing equations

$$\left(\frac{\partial \varrho}{\partial t} + \vec{u} \cdot \nabla \varrho \right) + \varrho \; \nabla \cdot \vec{u} = 0 \tag{34}$$

$$\varrho \; \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = \nabla \cdot \left(\mu \; \nabla \vec{u}\right) + \frac{1}{3}\nabla\left(\mu \; \nabla \cdot \vec{u}\right) - \nabla p + \varrho \;\; \vec{g} \tag{35}$$

$$\varrho \; c_p \; \left(\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T\right) = \nabla \cdot \left(\kappa \; \nabla T\right) + \beta T \left(\frac{\partial p}{\partial t} + \vec{u} \cdot \nabla p\right) + \Phi \tag{36}$$

The **Boussinesq's approximation** is applied under the hypotheses that the variations of pressure and temperature are limited and restrained around a reference state (T_0, p_0):

$p_0 = P_{atm}$ and $T_0 = \dfrac{T_W + T_J}{2}$.

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The **Boussinesq's approximation** is applied under the hypotheses that the variations of pressure and temperature are limited and restrained around a reference state (T_0, p_0):

$p_0 = P_{atm}$ and $T_0 = \dfrac{T_W + T_J}{2}$.

- All physical properties are evaluated at the reference state, with the only exception of density ϱ in the gravitational term:

  $$\varrho = \varrho_0; \quad \mu = \mu_0; \quad c_p = c_{p_0}; \quad \kappa = \kappa_0; \quad \nu = \nu_0 = \frac{\mu_0}{\varrho_0}; \quad \alpha = \alpha_0 = \frac{\kappa_0}{\varrho_0 c_{p_0}}.$$

- The density $\varrho(p, T)$, in the gravitational term, is replaced by the Boussinesq's density

  $$\varrho_{Bo} = \varrho_0 - \varrho_0\beta_0(T - T_0),\ \text{where}\ \beta = -\frac{1}{\varrho}\left(\frac{\partial\varrho}{\partial T}\right)_p \overset{\substack{perfect\\gas}}{=} \frac{1}{T} \overset{Boussinesq}{=} \frac{1}{T_0} = \beta_0$$

Buoyancy driven flow governing equations

$$\left(\frac{\partial \varrho_0}{\partial t} + \vec{u}\cdot\nabla\varrho_0\right) + \varrho_0\nabla\cdot\vec{u} = 0 \tag{34}$$

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Buoyancy driven flow governing equations

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Buoyancy driven flow governing equations

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Buoyancy driven flow governing equations

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- The gravitational terms are combined together and rewritten as $-\varrho_0 \beta_0 (T - T_0) \vec{g}$.

Buoyancy driven flow governing equations

$$\left(\frac{\partial \varrho_0}{\partial t} + \vec{u} \cdot \nabla \varrho_0\right) + \varrho_0 \nabla \cdot \vec{u} = 0 \quad (34)$$

$$\varrho_0 \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = \nabla \cdot (\mu_0 \nabla \vec{u}) + \frac{1}{3} \nabla (\mu_0 \nabla \cdot \vec{u}) - \nabla p' - \varrho_0 \beta_0 (T - T_0) \vec{g} \quad (35)$$

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Buoyancy driven flow governing equations (2)

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$$\varrho_0 \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = \nabla \cdot (\mu_0 \nabla \vec{u}) + \frac{1}{3} \nabla (\mu_0 \nabla \cdot \vec{u}) - \nabla p' - \varrho_0 \beta_0 (T - T_0) \vec{g} \tag{38}$$

$$\varrho_0 c_{p_0} \left(\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T\right) = \nabla \cdot (\kappa_0 \nabla T) + \beta_0 T \left(\frac{\partial p}{\partial t} + \vec{u} \cdot \nabla p\right) + \Phi \tag{39}$$

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- The unsteady and the advective terms of the continuity equations are indepedently null: $\frac{\partial \varrho_0}{\partial t} = 0; \quad \frac{\partial \varrho_0}{\partial x} = \frac{\partial \varrho_0}{\partial y} = 0; \quad$ (*material incompressibility*)

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- The steady state of the governing equations requires all the time derivatives to be null.

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- The pressure advection and the viscous dissipation terms are negligible, as hypotheses.

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- The steady state of the governing equations requires all the time derivatives to be null.
- The pressure advection and the viscous dissipation terms are negligible, as hypotheses.
- ...finally the **Boussinesq's equations** are found...

The Boussinesq's equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{40}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu_0\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{1}{\varrho_0}\frac{\partial p'}{\partial x} \tag{41}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu_0\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{1}{\varrho_0}\frac{\partial p'}{\partial y} - \varrho_0\beta_0(T - T_0)g \tag{42}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_0\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{43}$$

In order to identify the validity region of the Boussinesq's approximation, we have to validate the consistency of the following hypotheses :

The Boussinesq's equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \tag{40}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu_0\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{1}{\varrho_0}\frac{\partial p'}{\partial x} \tag{41}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = \nu_0\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{1}{\varrho_0}\frac{\partial p'}{\partial y}-\varrho_0\beta_0(T-T_0)g \tag{42}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \alpha_0\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{43}$$

In order to identify the validity region of the Boussinesq's approximation, we have to validate the consistency of the following hypotheses :

- the substitution of $\varrho(p,T)=\varrho_0+\varrho'+\varrho_h=\varrho_0[1+(\chi p'-\beta T')+\chi p_h]$ with the constant density ϱ_0;

The Boussinesq's equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{40}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu_0\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{1}{\varrho_0}\frac{\partial p'}{\partial x} \tag{41}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu_0\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{1}{\varrho_0}\frac{\partial p'}{\partial y} - \varrho_0\beta_0(T - T_0)g \tag{42}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_0\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{43}$$

In order to identify the validity region of the Boussinesq's approximation, we have to validate the consistency of the following hypotheses :

- the substitution of $\varrho(p,T) = \varrho_0 + \varrho' + \varrho_h = \varrho_0[1 + (\chi p' - \beta T') + \chi p_h]$ with the constant density ϱ_0;
- the neglecting of the pressure advection and the viscous dissipation terms.

We obtain 4 conditions (see rapport for passages):

a: $\boxed{\dfrac{\Delta T}{T_0} \ll 1}$ **b**: $\boxed{\dfrac{L}{T_0} \ll \dfrac{\overline{R}}{g}}$ **c**: $\boxed{\dfrac{L}{T_0} \ll \dfrac{c_{p0}}{g}}$ **d**: $\boxed{\dfrac{L}{T_0} \ll \dfrac{\Delta T}{T_0}\dfrac{c_{p0}}{g}}$

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- **d**: $\boxed{\frac{L}{T_0} \ll \frac{\Delta T}{T_0}\frac{c_{p0}}{g}}$
  The hydrostatic effects on temperature field are negligible (condition on the minumum vertical temperature gradient).

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